

A note on generalized S -space-forms

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Abstract. It is shown that, in a generalized S -space-form, F_7 always equals F_8 .

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1 Introduction

A $(2m + s)$ -dimensional Riemannian manifold (M, g) endowed with an f -structure f (that is, a tensor field of type $(1, 1)$ and rank $2m$ satisfying $f^3 + f = 0$ [4]) is said to be a *metric f -manifold* if there exist s global vector fields ξ_1, \dots, ξ_s on M (called *structure vector fields*) such that, if η_1, \dots, η_s are the dual 1-forms of ξ_1, \dots, ξ_s , then

$$f\xi_\alpha = 0; \quad \eta_\alpha \circ f = 0; \quad f^2 = -I + \sum_{\alpha=1}^s \eta_\alpha \otimes \xi_\alpha,$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^s \eta_\alpha(X)\eta_\alpha(Y)$$

for all $X, Y \in \mathcal{X}(M)$ and $\alpha = 1, \dots, s$. Let F be the fundamental 2-form on M defined by $F(X, Y) = g(X, fY)$ for any $X, Y \in \mathcal{X}(M)$. A metric f -manifold is *normal* if the Nijenhuis tensor $[f, f]$ of f equals $-2 \sum_{\alpha=1}^s \xi_\alpha \otimes d\eta_\alpha$. A normal metric f -manifold is said to be an *S -manifold* if $F = d\eta_\alpha$ for each $\alpha \in \{1, \dots, s\}$, while a normal metric f -manifold is a *K -manifold* if $dF = 0$. A K -manifold is a *C -manifold* if $d\eta_\alpha = 0$ for each $\alpha \in \{1, \dots, s\}$. For $s = 0$, a K -manifold is a Kaehler manifold and, for $s = 1$, a K -manifold is a quasi-Sasakian manifold, an S -manifold is a Sasakian manifold and a C -manifold is a cosymplectic manifold. For more details we refer to [1, 2].

A plane section Π on a metric f -manifold M is said to be an *f -section* if it is determined by a unit vector X orthogonal to all structure vector fields and the unit vector field fX . The sectional curvature of Π is called an *f -sectional curvature*. An S -manifold (resp., a C -manifold) is said to be an *S -space-form* (resp., a *C -space-form*) if it has constant f -sectional curvature c and then, it is denoted by $M(c)$ [3].

As a generalization of S -space-forms and C -space-forms, in [2] the authors introduced the concept of a generalized S -space-form equipped with two structure vector fields and studied its basic geometric properties. In fact, a metric f -manifold $(M, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ with two structure vector fields ξ_1 and ξ_2 is called a *generalized S -space-form* [2] if there exist differentiable functions F_1, \dots, F_8 on M such that the curvature tensor field R of M

satisfies

$$\begin{aligned}
R(X, Y)Z &= F_1 \{g(Y, Z)X - g(X, Z)Y\} \\
&+ F_2 \{g(X, fZ)fY - g(Y, fZ)fX + 2g(X, fY)fZ\} \\
&+ F_3 \{\eta_1(X)\eta_1(Z)Y - \eta_1(Y)\eta_1(Z)X + g(X, Z)\eta_1(Y)\xi_1 - g(Y, Z)\eta_1(X)\xi_1\} \\
&+ F_4 \{\eta_2(X)\eta_2(Z)Y - \eta_2(Y)\eta_2(Z)X + g(X, Z)\eta_2(Y)\xi_2 - g(Y, Z)\eta_2(X)\xi_2\} \\
&+ F_5 \{\eta_1(X)\eta_2(Z)Y - \eta_1(Y)\eta_2(Z)X + g(X, Z)\eta_1(Y)\xi_2 - g(Y, Z)\eta_1(X)\xi_2\} \\
&+ F_6 \{\eta_2(X)\eta_1(Z)Y - \eta_2(Y)\eta_1(Z)X + g(X, Z)\eta_2(Y)\xi_1 - g(Y, Z)\eta_2(X)\xi_1\} \\
&+ F_7 \{\eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1\} \\
&+ F_8 \{\eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2\}
\end{aligned} \tag{1.1}$$

for all $X, Y, Z \in \mathcal{X}(M)$.

Here, we prove the following

Theorem 1.1 *Let $(M, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ be a generalized S -space-form. Then $F_7 = F_8$.*

2 Proof of Theorem 1.1

From (1.1) we obtain

$$R(\xi_1, \xi_2)\xi_1 = -F_1\xi_2 + F_3\xi_2 + F_4\xi_2 - F_8\xi_2, \tag{2.1}$$

$$R(\xi_1, \xi_2)\xi_2 = F_1\xi_1 - F_3\xi_1 - F_4\xi_1 + F_7\xi_1. \tag{2.2}$$

Using (2.1) and (2.2) in

$$g(R(\xi_1, \xi_2)\xi_1, \xi_2) + g(R(\xi_1, \xi_2)\xi_2, \xi_1) = 0$$

we get $F_7 = F_8$. ■

References

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